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Nonlinear ship-wave theories by continuous mapping

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The exact equations of steady inviscid flow past a ship hull are formulated in a reference domain, onto which the flow domain is mapped. A thin-ship perturbation analysis is performed in the reference domain, and the first- and second-order solutions are derived. The classical thin-ship theory is obtained as the consistent, mapping-independent, perturbation solution in the physical space. Guilloton's method is interpreted as an inconsistent, mapping-dependent, second-order approximation. A new inconsistent approximation is obtained by exploiting the freedom in the mapping of the flow domain onto the reference domain. Further improvements are suggested.

1. Introduction

The analytical theory of the wave resistance of ships was dominated in the first half of the century by Michell's (1898) thin-ship theory. Unsatisfactory agreement between this linearized theory and the measured wave drag on hulls of practical shapes on the one hand and the advent of electronic computers on the other have stimulated interest in further development of the theory into the nonlinear range.

The Michell approximation may be obtained as the first-order solution in a systematic perturbation expansion of the velocity field in terms of the beam/length ratio as a small parameter, as shown apparently for the first time by Peters & Stoker (1957). The same approach may be employed to derive the equations satisfied by the second-order nonlinear term of the velocity expansion, a step which has been taken in somewhat different ways by Wehausen (1963) and Maruo (1966) (for a detailed review see Wehausen 1973).

The derivation of Michell's theory, as well as of the higher-order approximations, is based on a transfer of the boundary conditions at the free surface and on the hull by Taylor expansions about the undisturbed free surface and the ship centre-plane, respectively. Joseph (1973), in a discussion of the theory of two-dimensional progressive waves, finds the transfer of the velocity field from the actual flow domain beneath the free surface to the domain beneath the plane of the undisturbed free surface difficult to understand and suggests an alternative interpretation of the theory by continuous mapping. The difficulty underlined by

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Joseph (1973) is even more conspicuous in the case of three-dimensional flow past thin ships. For instance, the velocity at the intersection of the free surface and the hull is simultaneously projected by the Taylor expansions onto the unperturbed free surface and the centre-plane, which led to the belief that a second-order line-integral correction term was needed. It is one of our purposes in this paper to show that the higher-order thin-ship theory can be rigorously derived along Joseph's lines. We use, however, a more general approach in that perturbation expansions are not assumed from the outset and, more significant, the horizontal mapping is left arbitrary. The latter point has major consequences for the derivation of nonlinear theories other than the classical one.

The second-order term in the classical, consistent thin-ship expansion is extremely difficult to compute. Moreover, the expansion becomes non-uniform at small Froude numbers, as shown for the two-dimensional case by Salvesen (1969) and Dagan (1975*b*). These difficulties have prompted the seeking of simpler, approximate nonlinear corrections, one line of attack being the replacement of the Michell centre-plane singularities by modified ones, such that the associated linearized velocity field should be closer to an exact solution of the nonlinear problem. The most successful method of this type is apparently that of Guilloton (1964), who has suggested replacing the actual hull by a related 'linearized hull'. Wehausen (1969) has arrived at similar results as part of a complete second-order solution based on the use of Lagrangian co-ordinates. The apparent success of Guilloton's method in predicting ship-wave resistance better than Michell's theory (Gadd 1973) has stimulated attempts to arrive at rational derivations of the method by Noblesse (1975*a*) and Dagan (1975*a*), who independently suggested similar rationalizations of Guilloton's method by mapping the flow domain onto a reference domain and straining the co-ordinates. It was shown that Guilloton's method could be interpreted as a second-order inconsistent solution where the free-surface and hull boundary conditions were satisfied to second order, while the second-order field equations were ignored. Here the approach used by Noblesse and Dagan is further investigated and generalized, and, in particular, the field equations, as well as the boundary conditions, are solved consistently to second order. In addition, the use of the mapping for the purpose of simplifying the second-order nonlinear problem is explored in a more general way.

The main purpose of the present study is then to show that various nonlinear theories of flow past ship hulls, presented in the past somewhat disparately, can be derived by the fundamental approach of continuous mapping and perturbation expansions. In particular, a precise interpretation of the method of Guilloton is obtained. Furthermore, a new approximate solution generalizing Guilloton's method is presented, and a line of attack for further improvements is suggested.

2. The exact equations of flow past a ship hull

2.1. *Statement of the problem in physical space*

In this paper, we are concerned with steady free-surface gravity flow past a ship in an oncoming uniform stream of an inviscid incompressible fluid. The flow domain is assumed to be of infinite depth and infinite lateral extent.

Variables are made dimensionless with respect to the velocity $\mathscr{U}$ of the oncoming uniform stream, the acceleration of gravity g and the fluid density ρ, e.g. $\mathbf{X} = \mathbf{X}'g/\mathscr{U}^2$, $\mathbf{U} = \mathbf{U}'/\mathscr{U}$ and $R = R'g^2/\rho\mathscr{U}^6$, where $\mathbf{X}'(X', Y', Z')$ is the position vector of a point in the flow domain, $\mathbf{U}'(U', V', W')$ is the disturbance velocity (the components of the dimensionless total fluid velocity are then $1+U$, V and W) and R' is the wave resistance. The Cartesian system of co-ordinates $\mathbf{X}(X, Y, Z)$ is defined as follows: the unperturbed free surface and the ship centre-plane (which is a plane of symmetry for the flow) are taken as the planes $Y = 0$ and $Z = 0$, respectively, and the Y and X axes point upwards and towards the stern of the ship, respectively.

The hull of the ship is defined by the equation $Z = \pm F(X, Y)$. The hull function $F(X, Y)$ embodies the effects of the sinkage and trim experienced by the ship, and is therefore not known exactly beforehand. For simplicity, the analysis of the hydrodynamic problem is developed for a supposedly known hull and the effects of sinkage and trim are incorporated into the solution in the manner indicated in appendix C. The equation of the free surface is taken as $Y = E(X,Z)$. The flow domain is then defined by $-\infty < X < +\infty$, $Y \leqslant E(X,Z)$ and $|Z| \geqslant F(X, Y)$.

The exact equations and boundary conditions of the problem are well known (see, for instance, Wehausen & Laitone 1960, p. 447):

$$\nabla . \mathbf{U} = \nabla \times \mathbf{U} = 0 \quad \text{(in the flow domain)}, \tag{2.1}$$

$$W = \pm[(1+U)F_X + VF_Y] \quad \text{(on the ship hull)}, \tag{2.2}$$

$$\left.\begin{aligned} V &= (1+U)E_X + WE_Z \quad &(2.3)\\ U+\tfrac{1}{2}(U^2+V^2+W^2)+E &= 0 \quad &(2.4)\end{aligned}\right\} \text{(on the free surface).}$$

In addition, the radiation condition must be satisfied.

2.2. *Mapping of the flow domain onto a reference domain*

The complicated and unknown shape of the free surface suggests mapping the flow domain onto a simple, well-defined reference domain. A simple domain,† suggested by the thin-ship approximation, is the lower half-space bounded by $Y = 0$ and cut along a slit σ in the centre-plane $Z = 0$; see figure 1. It is emphasized that the slit σ need not, and in general will not, coincide with the projection

† Other, more complicated, reference domains, e.g. the domain bounded by $Y = 0$ and the hull $Z = \pm F(X, Y)$, could be considered. Other simple mappings could also easily be conceived, for instance, both the ship hull and the free surface could be mapped onto the undisturbed free surface $y = 0$; this would obviously lead to the so-called flat-ship theory. The slender-ship approximation, on the other hand, implies a degenerate mapping of the hull surface onto the line $y = z = 0$.

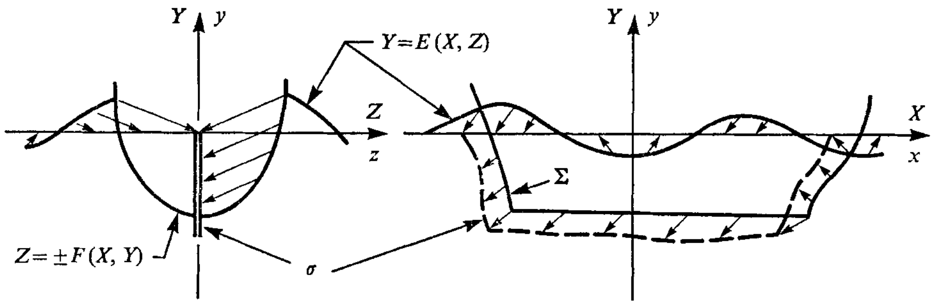

FIGURE 1. Mapping of the flow domain onto the reference domain.

Σ of the portion of the hull beneath $Y = 0$ onto the centre-plane. In fact, it is advantageous to leave the exact shape of σ unspecified at present.

The mapping is required to be one-to-one and to carry points on the actual free surface onto the undisturbed free surface $Y = 0$ and points on the hull onto points on the slit σ in the centre-plane (figure 1). The mapping is expressed by the equations

$$\mathbf{X} = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}), \quad \mathbf{U}(\mathbf{X}) = \mathbf{u}(\mathbf{x}), \qquad (2.5), (2.6)$$

where $\mathbf{x}(x,\ y,\ z)$ is the position vector of a point in the reference domain, $\boldsymbol{\xi}(\xi, \eta, \zeta)$ is the mapping vector and $\mathbf{u}(u, v, w)$ is the disturbance velocity expressed as a function of $\mathbf{x}$.

The free-surface mapping is expressed by (2.5) as

$$X = x + \xi(x, 0, z), \quad Y = E(X, Z) = \eta(x, 0, z), \quad Z = z + \zeta(x, 0, z) \qquad (2.7)$$

while the hull mapping implies

$$X = x + \xi(x, y, 0), \quad Y = y + \eta(x, y, 0), \quad Z = \pm F(X, Y) = \zeta(x, y, \pm 0), \quad (2.8)$$

where ξ and η are continuous across the slit σ but ζ undergoes a jump there.

Let us introduce the function $f(x, y)$ defined by

$$f(x, y) = F\{x + \xi(x, y, 0),\ y + \eta(x, y, 0)\}. \qquad (2.9)$$

The surface defined by $z = \pm f(x, y)$ has the same offsets as the ship hull $Z = \pm F(X, Y)$ but at 'displaced points' (x, y) rather than (X, Y). This surface corresponds to the 'linearized hull' introduced by Guilloton, and it is referred to as such below. By using (2.9), (2.8) gives

$$\zeta(x, y, \pm 0) = \pm f(x, y). \qquad (2.10)$$

Similar ideas, of mapping the flow domain onto a reference domain, underlie the articles by Yim (1968), Wehausen (1969), Landweber (1973), Joseph (1973), Dagan (1975*a*) and Noblesse (1975*a*). However, the approach presented above and in § 2.3, following Noblesse (1975*b*), is more general.

Naturally, this idea of mapping physical space onto a simple reference domain has long ago been exploited in the theory of two-dimensional free-surface flows, where the complex potential is extensively used as the independent variable. One should realize, however, that the remarkable properties associated with conformal mappings do not carry over to the three-dimensional case.

2.3. *Statement of the problem in the reference space*

The (exact) equations in the reference space (x, y, z) may be obtained by introducing the mapping (2.5), (2.6) into (2.1)–(2.4). Whenever it is convenient, we shall use the following tensor notation: X_i, U_i, x_i, ξ_i and u_i $(i = 1, 2, 3)$ denote the components of the vectors $\mathbf{X}$, $\mathbf{U}$, $\mathbf{x}$, $\boldsymbol{\xi}$ and $\mathbf{u}$, and $U_{i,j} \equiv \partial U_i/\partial X_j$, $u_{i,j} \equiv \partial u_i/\partial x_j$ and $\xi_{i,j} \equiv \partial \xi_i/\partial x_j$ represent the components of the gradient tensors $\nabla\mathbf{U}$, $\nabla\mathbf{u}$ and $\nabla\boldsymbol{\xi}$.

Equation (2.6) yields by differentiation

$$U_{i,k}(\mathbf{X})\, dX_k = u_{i,j}(\mathbf{x})\, dx_j \quad (i = 1, 2, 3),$$

where the summation convention applies, here and hereafter. By using (2.5), we obtain

$$U_{i,k}(\mathbf{X})\, [\delta_{kj} + \xi_{k,j}(\mathbf{x})] = u_{i,j}(\mathbf{x}) \quad (i, j = 1, 2, 3), \tag{2.11}$$

where δ_{kj} is the Kronecker delta. Solving the linear system (2.11) results in

$$U_{i,k}(\mathbf{X}) = u_{i,j}(\mathbf{x})\, a_{jk}(\mathbf{x}) \quad (i, k = 1, 2, 3),$$

where a_{jk} denotes the inverse of the matrix $\delta_{kj} + \xi_{k,j}$. By substituting into (2.1), we obtain the transformed field equations

$$u_{i,j}\, a_{ji} = \epsilon_{ijk} u_{k,l} a_{lj} = 0 \quad (-\infty < x < +\infty,\ y < 0,\ |z| > 0), \tag{2.12}$$

where ϵ_{ijk} is the permutation tensor.

Equation (2.8) yields

$$dX = (1 + \xi_x)\, dx + \xi_y\, dy, \quad dY = \eta_x\, dx + (1 + \eta_y)\, dy \quad (z = 0),$$
$$\zeta_x\, dx + \zeta_y\, dy = \pm (F_X\, dX + F_Y\, dY) \quad (z = \pm 0).$$

Eliminating dX and dY, solving for F_X and F_Y, substituting into (2.2) and using (2.6) yields (for details, see Noblesse 1975*b*) the transformed hull condition

$$w(1 + \xi_x + \eta_y + \xi_x \eta_y - \xi_y \eta_x) = (1 + u)(\zeta_x + \zeta_x \eta_y - \zeta_y \eta_x) + v(\zeta_y + \zeta_y \xi_x - \zeta_x \xi_y) \quad (z = \pm 0). \tag{2.13}$$

The kinematic free-surface condition (2.3) can similarly be shown to become

$$v(1 + \xi_x + \zeta_z + \xi_x \zeta_z - \xi_z \zeta_x) = (1 + u)(\eta_x + \eta_x \zeta_z - \eta_z \zeta_x) + w(\eta_z + \eta_z \xi_x - \eta_x \xi_z) \quad (y = 0), \tag{2.14}$$

and it is readily seen that the dynamic free-surface condition (2.4) becomes

$$u + \tfrac{1}{2}(u^2 + v^2 + w^2) + \eta = 0 \quad (y = 0). \tag{2.15}$$

2.4. *Summary*

The exact equations of flow past a ship hull have been formulated in the physical space (X, Y, Z) [(2.1)–(2.4)] and, by mapping the flow domain onto a reference domain (2.5), (2.6), in the reference space (x, y, z) [(2.12)–(2.15)]. One difference between the two formulations lies in the fact that the domain where the solution is sought is complicated and unknown in the physical-space formulation, but becomes simple and known beforehand in the formulation in the reference space. A more significant difference, perhaps, is to be found in the fact that the solution is sought in explicit form, i.e. $\mathbf{U}(\mathbf{X})$ and $Y = E(X, Z)$, in the usual physical-space formulation, whereas in the reference-space formulation the solution is sought in the parametric form (2.5) and (2.6). An interesting feature of parametric representations is that they are not unique, and hence offer some degree of arbitrariness. Indeed, the mapping of the flow domain onto the reference domain, and hence the mapping vector $\boldsymbol{\xi}(\mathbf{x})$, is arbitrary except for the requirement that the ship hull and free surface be mapped onto the ship centre-plane and undisturbed free surface respectively (for instance, ξ could clearly be set equal to zero).

A common feature of both formulations, however, is that they are nonlinear and appear to be equally intractable. We thus proceed with a thin-ship perturbation analysis.

3. Thin-ship perturbation analysis in the reference space

In this section, a thin-ship regular perturbation analysis of the foregoing exact nonlinear problem is carried out in the reference space, and the first- and second-order problems are formulated and solved.

3.1. *Perturbation expansions*

Let ϵ be the beam/length ratio, and let the function $F^{(1)}(X, Y)$ be defined by

$$F(X, Y) = \epsilon F^{(1)}(X, Y). \tag{3.1}$$

The parameter ϵ is assumed to be small, and an asymptotic solution is sought by means of the following asymptotic expansions:

$$f(x,y) = \epsilon f^{(1)}(x,y) + \epsilon^2 f^{(2)}(x,y) + \ldots, \tag{3.2}$$

$$\mathbf{u}(\mathbf{x}) = \epsilon \mathbf{u}^{(1)}(\mathbf{x}) + \epsilon^2 \mathbf{u}^{(2)}(\mathbf{x}) + \ldots, \tag{3.3}$$

$$\boldsymbol{\xi}(\mathbf{x}) = \epsilon \boldsymbol{\xi}^{(1)}(\mathbf{x}) + \epsilon^2 \boldsymbol{\xi}^{(2)}(\mathbf{x}) + \ldots. \tag{3.4}$$

Substituting (3.1) and (3.2) into (2.9) yields

$$\begin{aligned} &\epsilon f^{(1)}(x,y) + \epsilon^2 f^{(2)}(x,y) + \ldots \\ &\qquad = \epsilon F^{(1)}\{x + \epsilon \xi^{(1)}(x,y,0) + \ldots,\ y + \epsilon \eta^{(1)}(x,y,0) + \ldots\}, \end{aligned} \tag{3.5}$$

where $(x,y) \in \sigma$. This equation can be used to express the 'linearized hull' function $f(x,y)$ in terms of the ship hull $F^{(1)}(X, Y)$ either explicitly by expanding the right-hand side of (3.5) in a Taylor series (as in § 4), or implicitly via an iterative procedure† (as in § 5). However, we are not concerned with these alternatives in the present section. The object of this section is to derive the first- and second-order perturbation solutions in the reference space in the most general form by using the general hull expansion (3.2).

The first- and second-order approximations to the field equations may be derived from (2.12) by substituting expansions (3.3) and (3.4). It is simpler, however, to proceed from (2.1) and (2.11), which can be rewritten as

$$U_{i,j}(\mathbf{X}) = u_{i,j}(\mathbf{x}) - U_{i,k}(\mathbf{X})\, \xi_{k,j}(\mathbf{x}) \quad (i,j = 1,2,3). \tag{3.6}$$

† In the latter case, $f^{(1)}(x,y)$, $f^{(2)}(x,y), \ldots$ depend on ϵ, i.e. we have $f^{(1)}(x,y;\epsilon)$, $f^{(2)}(x,y;\epsilon), \ldots$.

3.2. *The first-order problem*

By using (3.6), (3.3), (2.5) and (3.4), we readily obtain for the first-order approximation to the field equations (2.1)

$$\nabla \cdot \mathbf{u}^{(1)}(\mathbf{x}) = \nabla \times \mathbf{u}^{(1)}(\mathbf{x}) = 0 \quad (y < 0). \tag{3.7 a, b}$$

Equations (2.10), (3.2) and (3.4) give

$$\zeta^{(1)}(x, y, \pm 0) = \pm f^{(1)}(x, y) \quad ((x, y) \in \sigma). \tag{3.8}$$

By substituting (3.3), (3.4) and (3.8) into (2.13), we obtain

$$w^{(1)}(x, y, \pm 0) = \pm f^{(1)}_x(x, y) \quad ((x, y) \in \sigma). \tag{3.9}$$

Similarly, substituting (3.3) and (3.4) into (2.14) and (2.15) yields

$$v^{(1)} = \eta^{(1)}_x, \quad \eta^{(1)} = -u^{(1)} \quad (y = 0). \tag{3.10), (3.11}$$

Equations (3.7), (3.9), (3.10) and (3.11) may be recognized as the equations of Michell's approximation. Equations (3.7) give

$$\mathbf{u}^{(1)} = \nabla \phi^{(1)}, \tag{3.12}$$

$$\nabla^2 \phi^{(1)} = 0 \quad (y < 0). \tag{3.13}$$

Substituting (3.12) into (3.9), (3.10) and (3.11) yields

$$\phi^{(1)}_z(x, y, \pm 0) = \pm f^{(1)}_x(x, y) \quad ((x, y) \in \sigma), \tag{3.14}$$

$$\phi^{(1)}_{xx} + \phi^{(1)}_y = 0 \quad (y = 0). \tag{3.15}$$

The velocity potential $\phi^{(1)}$ satisfying (3.13), the boundary conditions (3.14) and (3.15) and the radiation condition is given by

$$\phi^{(1)}(x, y, z) = \frac{1}{2\pi} \iint_\sigma G(x, y, z;\, x', y', 0) f^{(1)}_x(x', y')\, dx'\, dy', \tag{3.16}$$

where $G(x, y, z;\, x', y', z')$ is the Havelock source potential (see, for instance, Wehausen & Laitone 1960, pp. 579 and 484).

The only restrictions imposed upon the first-order mapping $\boldsymbol{\xi}^{(1)}(\mathbf{x})$ are (3.8) and (3.11). Therefore $\xi^{(1)}$, as well as the extensions of $\eta^{(1)}$ and $\zeta^{(1)}$ to $y < 0$ and $|z| > 0$, respectively, are unspecified and may be chosen arbitrarily, as noted in § 2.4.

3.3. *The second-order problem*

The second-order field equations are likewise obtained by substituting (3.6), (3.3), (2.5) and (3.4) into (2.1). This yields

$$\left.\begin{aligned} u^{(2)}_{i,i} &= u^{(1)}_{i,k}\, \xi^{(1)}_{k,i} && (3.17)\\ u^{(2)}_{i,j} - u^{(2)}_{j,i} &= u^{(1)}_{i,k}\, \xi^{(1)}_{k,j} - u^{(1)}_{j,k}\, \xi^{(1)}_{k,i} \quad (i \neq j) && (3.18) \end{aligned}\right\} \quad (y < 0).$$

Equations (2.10), (3.2) and (3.4) yield

$$\zeta^{(2)}(x, y, \pm 0) = \pm f^{(2)}(x, y) \quad ((x, y) \in \sigma). \tag{3.19}$$

Substituting (3.3), (3.4) and (3.19) into (2.13) gives the hull condition

$$w^{(2)}(x,y,\pm 0) = \pm[f_x^{(2)} + (u^{(1)} - \xi_x^{(1)})f_x^{(1)} + (v^{(1)} - \eta_x^{(1)})f_y^{(1)}] \quad ((x,y)\in\sigma), \tag{3.20}$$

where (3.8) and (3.9) have been used. The free-surface conditions are similarly obtained by substituting (3.3) and (3.4) into (2.14) and (2.15) and using (3.10). We obtain

$$\left.\begin{aligned} v^{(2)} &= \eta_x^{(2)} + u^{(1)}\eta_x^{(1)} + w^{(1)}\eta_z^{(1)} - v^{(1)}\xi_x^{(1)} - \eta_z^{(1)}\zeta_x^{(1)} \quad &(3.21)\\ \eta^{(2)} &= -u^{(2)} - \tfrac{1}{2}(u^{(1)2} + v^{(1)2} + w^{(1)2}) \quad &(3.22)\end{aligned}\right\} \quad (y=0).$$

Eliminating $\eta^{(2)}(x,0,z)$ between (3.21) and (3.22) yields

$$u_x^{(2)} + v^{(2)} = -\tfrac{1}{2}(u^{(1)2} + v^{(1)2} + w^{(1)2})_x + (\xi_x^{(1)} - u^{(1)})u_x^{(1)} + (\zeta_x^{(1)} - w^{(1)})u_z^{(1)} \quad (y=0), \tag{3.23}$$

where (3.10) and (3.11) have been used.

The general solution† of the field equations (3.17) and (3.18) is found by inspection of the terms on the right-hand side to be given by

$$\mathbf{u}^{(2)}(\mathbf{x}) = (\boldsymbol{\xi}^{(1)}.\nabla)\mathbf{u}^{(1)} + \nabla\phi^{(2)}, \tag{3.24}$$

where $\phi^{(2)}(\mathbf{x})$ satisfies the Laplace equation

$$\nabla^2\phi^{(2)} = 0 \quad (y<0). \tag{3.25}$$

Substituting (3.24) into (3.20), and using (3.8) and (3.7 *a*), yields the hull condition

$$\phi_z^{(2)}(x,y,\pm 0) = \pm\tau^{(2)}(x,y) = \pm[(u^{(1)}f^{(1)})_x + (v^{(1)}f^{(1)})_y + (f^{(2)} - \xi^{(1)}f_x^{(1)} - \eta^{(1)}f_y^{(1)})_x] \quad ((x,y)\in\sigma). \tag{3.26}$$

Similarly, substituting (3.24) into (3.23) and using (3.10), (3.11) and (3.7 *b*) gives the free-surface condition

$$\phi_{xx}^{(2)} + \phi_y^{(2)} = \pi^{(2)}(x,z) = -(u^{(1)2} + v^{(1)2} + w^{(1)2})_x + u^{(1)}(u_x^{(1)} + v^{(1)})_y \quad (y=0). \tag{3.27}$$

The second-order potential $\phi^{(2)}$ satisfying (3.25), (3.26), (3.27) and the radiation condition is given by

$$\phi^{(2)}(x,y,z) = \frac{1}{2\pi}\iint_\sigma G(x,y,z;x',y',0)\,\tau^{(2)}(x',y')\,dx'\,dy' - \frac{1}{4\pi}\iint_{\mathrm{FS}} G(x,y,z;x',0,z')\,\pi^{(2)}(x',z')\,dx'\,dz', \tag{3.28}$$

where FS (free surface) denotes the plane $y=0$. Again, the mapping $\boldsymbol{\xi}^{(2)}$ is arbitrary, except for (3.19) and (3.22).

3.4. *Summary*

A thin-ship perturbation solution has been obtained in the reference space. To second order, the velocity field is given by

$$\mathbf{u}(\mathbf{x}) = \epsilon\nabla\phi^{(1)} + \epsilon^2[(\boldsymbol{\xi}^{(1)}.\nabla)\nabla\phi^{(1)} + \nabla\phi^{(2)}] + O(\epsilon^3), \tag{3.29}$$

† An alternative derivation of (3.12), (3.13), (3.24) and (3.25), based on a Taylor-expansion approach along the lines of Joseph's (1973) study, may also be instructive and is given in appendix A.

where the first- and second-order potentials $\phi^{(1)}$ and $\phi^{(2)}$ are given by (3.16) and (3.28) in terms of 'centre-plane sources' of strength $f_x^{(1)}$ and $\tau^{(2)}$ [see (3.26)] and 'free-surface sources' $\pi^{(2)}$ [see (3.27)]. It is easy to ascertain [see equation (A 5) in appendix A] that the third-order approximation to the velocity is given by

$$\begin{aligned}\mathbf{u}(\mathbf{x}) = {} & \epsilon\nabla\phi^{(1)} + \epsilon^2[(\boldsymbol{\xi}^{(1)}.\nabla)\nabla\phi^{(1)} + \nabla\phi^{(2)}] \\ & + \epsilon^3\{[\tfrac{1}{2}(\boldsymbol{\xi}^{(1)}.\nabla)^2 + (\boldsymbol{\xi}^{(2)}.\nabla)]\nabla\phi^{(1)} + (\boldsymbol{\xi}^{(1)}.\nabla)\nabla\phi^{(2)} + \nabla\phi^{(3)}\} \\ & + O(\epsilon^4),\end{aligned} \tag{3.30}$$

where the third-order potential $\phi^{(3)}$ is given by (3.28) with the second-order sources $\tau^{(2)}$ and $\pi^{(2)}$ replaced by third-order sources $\tau^{(3)}$ and $\pi^{(3)}$, which can easily be obtained by substituting (3.30) into the hull and free-surface conditions (2.13) and (2.14), (2.15).

The solution in the physical space $\mathbf{X}$ is then determined by substituting (3.29) and (3.4) into (2.6) and (2.5), respectively. We thus obtain

$$\mathbf{U}(\mathbf{X}) = \epsilon\nabla\phi^{(1)}(\mathbf{x}) + \epsilon^2\{[\boldsymbol{\xi}^{(1)}(\mathbf{x}).\nabla]\nabla\phi^{(1)}(\mathbf{x}) + \nabla\phi^{(2)}(\mathbf{x})\} + O(\epsilon^3), \tag{3.31 a}$$

$$\mathbf{X} = \mathbf{x} + \epsilon\boldsymbol{\xi}^{(1)}(\mathbf{x}) + O(\epsilon^2). \tag{3.31 b}$$

Expansions (3.31 a, b) express the disturbance velocity $\mathbf{U}$ in parametric form, as noted in § 2.4. The main feature of this parametric representation of the perturbation solution is that the mapping vectors $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \ldots$ are arbitrary except for the requirements on the ship centre-plane [(3.8) and (3.19)] and on the free surface [(3.11) and (3.22)]. In the following section, it will be shown, however, that the consistent perturbation solution in physical space is independent of the mapping vectors, and that this consistent solution yields the classical thin-ship theory. The parametric form (3.31 a, b) of the perturbation solution will be explored in § 5, leading to an interpretation and generalization of the method of Guilloton (1964).

4. Consistent perturbation solution and the classical thin-ship theory

4.1. *Consistent perturbation solution in physical space*

A consistent expansion in physical space is defined as

$$\mathbf{U}(\mathbf{X}) = \epsilon\mathbf{U}^{(1)}(\mathbf{X}) + \epsilon^2\mathbf{U}^{(2)}(\mathbf{X}) + \ldots, \tag{4.1}$$

where ϵ does not appear in the functions $\mathbf{U}^{(n)}$, which therefore depend on $\mathbf{X}$ alone. Expanding (4.1) in a Taylor series about the point $\mathbf{x}$ in the reference space, using (3.31 b), grouping the terms of the same order in ϵ and comparing with (3.31 a) shows that we have

$$\mathbf{U}(\mathbf{X}) = \epsilon\nabla\phi^{(1)}(\mathbf{X}) + \epsilon^2\nabla\phi^{(2)}(\mathbf{X}) + \ldots \tag{4.2}$$

in agreement with equation (A 3) in appendix A. It is easily seen that expansion (4.2), to order ϵ^n, can be derived from expansions (3.31 a, b) by retaining terms up to order ϵ^n in (3.31 a) and ϵ^{n-1} in (3.31 b), and eliminating the parameter $\mathbf{x}$ and the mapping vectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(n-1)}$.

The perturbation solution (4.2) is not yet strictly consistent since the velocity potentials $\phi^{(1)}, \phi^{(2)}, \ldots$ are expressed in terms of the 'linearized hull functions' $f^{(1)}, f^{(2)}, \ldots$, which involve the mapping, and therefore ϵ. To obtain a consistent expansion we must also expand the hull function $F^{(1)}(X, Y)$ in (3.5) consistently in a Taylor series. This yields

$$f^{(1)}(x,y) = F^{(1)}(x,y), \tag{4.3}$$

$$f^{(2)}(x,y) = \xi^{(1)}(x,y,0)\,F_X^{(1)}(x,y)+\eta^{(1)}(x,y,0)\,F_Y^{(1)}(x,y), \tag{4.4}$$

where $(x,y)\in\Sigma$, i.e. σ is identical with Σ. Thus the consistent hull expansion implies a restriction on the mapping of the wetted part of the hull, namely that it be mapped onto the projection Σ of the portion of the hull beneath $y = 0$ onto the centre-plane.

The first-order potential $\phi^{(1)}$ is then obtained from (3.16) by replacing $f_x^{(1)}(x,y)$ by $F_x^{(1)}(x,y)$ and σ by Σ. Similarly, the second-order potential $\phi^{(2)}$ is given by (3.28) with σ and $\tau^{(2)}(x,y)$ replaced by Σ and $\tau_c^{(2)}(x,y)$, defined by

$$\tau_c^{(2)}(x,y) = [u^{(1)}(x,y,0)\,F^{(1)}(x,y)]_x + [v^{(1)}(x,y,0)\,F^{(1)}(x,y)]_y \quad ((x,y)\in\Sigma), \tag{4.5}$$

which readily follows from (4.3), (4.4) and (3.26).

Therefore, it may be seen that the consistent perturbation solution in physical space is independent of the mapping, which appears neither in the expressions for $\phi^{(1)}$ and $\phi^{(2)}$ nor in (4.2), and is thus unambiguously, and explicitly, defined in terms of the ship hull. The fact that the consistent perturbation solution in physical space is independent of the mapping was previously established by Joseph (1973) in his discussion of two-dimensional progressive waves.

Equation (4.2) is readily understandable when $\mathbf{X}$ is within the reference space, i.e. when $Y \leqslant 0$. However, when $\mathbf{X}$ is outside the reference space, i.e. when $0 < Y \leqslant E(X,Z)$, $\phi^{(1)}(\mathbf{X}), \phi^{(2)}(\mathbf{X}), \ldots$ must be understood as the analytic continuations of $\phi^{(1)}(\mathbf{X}), \phi^{(2)}(\mathbf{X}), \ldots$ above $Y = 0$, since the Green function $G(\mathbf{X},\mathbf{x}')$ is defined for $Y \leqslant 0$ only. We may also, in the spirit of the classical thin-ship theory, define the extension of $\phi^{(1)}(\mathbf{X})$, $\phi^{(2)}(\mathbf{X}), \ldots$ outside the reference space by means of a Taylor expansion about a point $\mathbf{x}$ inside the reference space. To first and second order, (4.2) may thus be written as†

$$\mathbf{U}(\mathbf{X}) = \epsilon\nabla\phi^{(1)}(\mathbf{x})+O(\epsilon^2), \tag{4.6a}$$

$$\mathbf{U}(\mathbf{X}) = [1+(\mathbf{X}-\mathbf{x})\,.\,\nabla]\,\epsilon\nabla\phi^{(1)}(\mathbf{x})+\epsilon^2\nabla\phi^{(2)}(\mathbf{x})+O(\epsilon^3), \tag{4.6b}$$

where $\mathbf{X}$ and $\mathbf{x}$ are any two arbitrary points in the physical and reference space with the only restriction that $|\mathbf{X}-\mathbf{x}| = O(\epsilon)$, i.e. that the distance between $\mathbf{X}$ and $\mathbf{x}$ be of order ϵ. It is emphasized that, while in (3.31 *a, b*) $\mathbf{X}$ is a continuous function of $\mathbf{x}$, no such continuous correspondence between $\mathbf{X}$ and $\mathbf{x}$ is implied in (4.6 *a, b*). Indeed, the values of $\mathbf{U}(\mathbf{X})$ obtained by using two different points $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $|\mathbf{x}_1-\mathbf{x}_2| = O(\epsilon)$ differ by terms of order ϵ^2 in (4.6 *a*) and ϵ^3 in (4.6 *b*), i.e. discontinuities of higher order.

Finally, the free surface is defined by

$$Y = -\epsilon\phi_X^{(1)}(X,0,Z)+\epsilon^2[\phi_X^{(1)}\phi_{XY}^{(1)}-\tfrac{1}{2}|\nabla\phi^{(1)}|^2-\phi_X^{(2)}]_{Y=0}+O(\epsilon^3), \tag{4.7}$$

which follows from (2.7), (3.4), (3.11), (3.12), (3.22) and (3.24).

† Equations (4.6 *a, b*) may also be obtained directly from (3.31 *a, b*) by eliminating the mapping vectors $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \ldots$.

4.2. *The classical thin-ship theory*

Equation (4.2), where the first- and second-order potentials $\phi^{(1)}$ and $\phi^{(2)}$ are given by (3.16) and (3.28) with $\sigma, f^{(1)}$ and $\tau^{(2)}$ replaced by Σ, $F^{(1)}$ and $\tau_c^{(2)}$ [see (4.5)], and the free-surface equation (4.7) may be recognized as the results of the classical thin-ship theory of Michell (1898) and, with somewhat different forms for the second-order terms,† of Wehausen (1963) and Maruo (1966).

Strictly speaking however, the classical thin-ship theory is to be recognized in (4.6), rather than (4.2), for points **X** outside the reference space, i.e. for $0 < Y \leqslant E(X, Z)$. In fact, in the classical derivation of the thin-ship theory, (4.6) is to be interpreted in an even more restricted sense, namely, the value of the velocity $\mathbf{U}(\mathbf{X})$ at a point **X** on the free surface is evaluated by taking the projection of **X** onto the undisturbed free surface $Y = 0$ as the point **x** in the Taylor series (4.6), while the value of $\mathbf{U}(\mathbf{X})$ at a point **X** on the ship hull is obtained from the projection **x** of **X** onto the ship centre-plane $Z = 0$. Thus, in the words of the discussion following (4.6), a discontinuous correspondence between **X** and **x** is implied in the classical derivation and interpretation of the thin-ship theory. This discontinuous correspondence is somewhat confusing, particularly at the intersection of the free surface and the hull, and has led to the belief that an additional line-integral term was needed.

Whereas the first-order potential $\phi^{(1)}$ can be computed relatively easily, the computation of the second-order potential $\phi^{(2)}$ poses a formidable problem, even in the case of simple analytical hull shapes. The main difficulty is associated with the free-surface source distribution $\pi^{(2)}$ [see (3.27)].

In addition to these numerical difficulties, the thin-ship consistent solution is non-uniform as the Froude number tends to zero, and the computation of the second-order term thus becomes of limited value in this limit. This problem was investigated by Dagan (1975*b*) in the case of two-dimensional flow past submerged bodies: it was shown that the non-uniformity depends essentially on the shape of the leading edge, and that it worsens as the shape becomes blunter. An estimate of the spectra of the free-waves generated by a wedge-like bow shape (Dagan 1973) shows that the ratio between the amplitude of the second- and first-order far free-waves is $O(\epsilon \ln F)$, where F is the Froude number based on ship length. Hence, for a ship of given beam/length ratio, the consistent thin-ship expansion becomes non-uniform as $F \to 0$. The remedy suggested in the two-dimensional case (Dagan 1975*b*) is to incorporate the second-order term into the first-order one by co-ordinate straining.

5. Mapping-dependent perturbation solutions and the method of Guilloton

5.1. *Mapping-dependent inconsistent second-order solution*

It was shown in the previous section that the consistent second-order solution in physical space could be obtained from expansions (3.31 *a*, *b*) by retaining terms

† *Note added in proof.* The second-order terms (3.28), (3.27) and (4.5) are in exact agreement with the results given in equations (15) and (12) of Eggers (1966).

(5.1 *a*, *b*) actually exact to second order. By comparing (5.1 *a*) with (3.31 *a*), it is immediately evident that this would be achieved by selecting $\boldsymbol{\xi}^{(1)}$ such that

$$[\boldsymbol{\xi}^{(1)}(\mathbf{x})\,.\,\nabla]\,\nabla\phi^{(1)}(\mathbf{x})+\nabla\phi^{(2)}(\mathbf{x}) = 0. \tag{5.5}$$

By using (5.1 *b*), (4.1) and (4.2), (5.5) may be written as

$$\mathbf{U}^{(1)}(\mathbf{X})-\mathbf{U}^{(1)}(\mathbf{x})+\mathbf{U}^{(2)}(\mathbf{x}) = 0,$$

which may be interpreted very simply as follows: can we find a continuous mapping $\mathbf{X}\leftrightarrow\mathbf{x}$ such that the difference between the first-order velocity $\mathbf{U}^{(1)}$ at points $\mathbf{X}$ and $\mathbf{x}$, i.e. $\mathbf{U}^{(1)}(\mathbf{X})-\mathbf{U}^{(1)}(\mathbf{x})$, cancels the second-order velocity $\mathbf{U}^{(2)}(\mathbf{x})$? At first glance it would seem unlikely that the answer to this question should be in the affirmative because $\eta^{(1)}$ and $\zeta^{(1)}$ are defined by (3.8) and (3.11) along $y = z = 0$, so that (5.5) becomes a system of three algebraic equations for only one unknown $\xi^{(1)}$.

In any case, the purpose of this subsection was mainly to show that the first-order approximation (5.1 *a*, *b*) offered the possibility of being improved, i.e. rendered a good approximation to the exact second-order solution, by exploiting the freedom in the mapping $\boldsymbol{\xi}^{(1)}$. Furthermore, this provided an interpretation of the method of Guilloton. In the following subsection, it will be shown that the arbitrariness in the mapping $\boldsymbol{\xi}^{(1)}$ may also be used advantageously for the purpose of simplifying the complete second-order solution.

5.2. *Derivation of a simplified second-order solution by using the first-order mapping*

We recall that in § 4 the complete second-order solution (3.31 *a*, *b*) was shown to be independent of the mapping $\boldsymbol{\xi}^{(1)}$ to order ϵ^2. In fact, $\boldsymbol{\xi}^{(1)}$ was eliminated and the solution (3.31 *a*, *b*) was written in explicit form (leading to the classical thin-ship theory). It is clear, however, that the mapping $\boldsymbol{\xi}^{(1)}$ need not be eliminated from (3.31 *a*, *b*). In this subsection the complete second-order approximation, in the parametric form (3.31 *a*, *b*), is reconsidered, and after rewriting $\mathbf{u}^{(2)}$ in (3.24) and (3.28) in an alternative form, the freedom in the selection of the mapping is exploited to derive a second-order solution circumventing the difficulty associated with the numerical evaluation of the contribution of the free-surface source density $\pi^{(2)}$ to the second-order potential $\phi^{(2)}$ [see (3.28)].

We first derive the alternative expression for the second-order velocity $\mathbf{u}^{(2)}$. By using (3.7 *b*), we observe that the irrotationality equations (3.18) are identically satisfied if $\mathbf{u}^{(2)}$ is written as

$$u_i^{(2)}(\mathbf{x}) = -u_j^{(1)}\xi_{j,i}^{(1)}+\theta_{,i}^{(2)}, \tag{5.6}$$

where $\theta^{(2)}$ is any arbitrary scalar function. Successively substituting (5.6) into the continuity equation (3.17), the hull condition (3.20) and the free-surface condition (3.23) yields

$$\nabla^2\theta^{(2)} = \delta^{(2)}(\mathbf{x}) = \nabla^2(\boldsymbol{\xi}^{(1)}\,.\,\mathbf{u}^{(1)}) \quad (y < 0), \tag{5.7}$$

$$\begin{aligned}\theta_z^{(2)}(x,y,\pm 0) = \pm\tilde{\tau}^{(2)}(x,y) = \pm\,[&f_x^{(2)}+(u^{(1)}-\xi_x^{(1)}+\zeta_z^{(1)})f_x^{(1)}\\ &+(v^{(1)}-\eta_x^{(1)})f_y^{(1)}+(u^{(1)}\xi_z^{(1)}+v^{(1)}\eta_z^{(1)})_{z=+0}] \quad ((x,y)\in\sigma),\end{aligned} \tag{5.8}$$

$$\begin{aligned}\theta_{xx}^{(2)}+\theta_y^{(2)} = \tilde{\pi}^{(2)}(x,z) = \mathbf{u}^{(1)}\,.\,&(\boldsymbol{\xi}_{xx}^{(1)}+\boldsymbol{\xi}_y^{(1)})\\ &+2(\boldsymbol{\xi}_x^{(1)}-\mathbf{u}^{(1)})\,.\,\mathbf{u}_x^{(1)}-(\eta_x^{(1)}-v^{(1)})\,v_x^{(1)} \quad (y = 0),\end{aligned} \tag{5.9}$$

where (3.9) and (3.7 *b*) have been used in (5.8) and (5.9) respectively. Equations (5.7)–(5.9) could also have been derived from (3.25)–(3.27) by using the relationship $\theta^{(2)} = \phi^{(2)} + \boldsymbol{\xi}^{(1)} . \mathbf{u}^{(1)}$, which readily follows from (3.24) and (5.6).

The solution $\theta^{(2)}$ satisfying the Poisson equation (5.7), the boundary conditions (5.8) and (5.9) and the radiation condition is given by

$$\begin{aligned} \theta^{(2)}(\mathbf{x}) = & \frac{1}{4\pi}\iiint G(\mathbf{x};\mathbf{x}')\,\delta^{(2)}(\mathbf{x}')\,d\mathbf{x}' \\ & + \frac{1}{2\pi}\iint_{\sigma} G(\mathbf{x};x',y',0)\,\tilde{\tau}^{(2)}(x',y')\,dx'\,dy' \\ & - \frac{1}{4\pi}\iint_{\mathrm{FS}} G(\mathbf{x};x',0,z')\,\tilde{\pi}^{(2)}(x',z')\,dx'\,dz'. \end{aligned} \tag{5.10}$$

Equations (5.6) and (5.10) define the second-order velocity $\mathbf{u}^{(2)}$ in terms of distributions of sources of strengths $\delta^{(2)}$, $\tilde{\tau}^{(2)}$ and $\tilde{\pi}^{(2)}$ over the lower half-space $y \leqslant 0$, the centre-plane $z = 0$ and the free surface $y = 0$, respectively.

The source densities $\delta^{(2)}$, $\tilde{\tau}^{(2)}$ and $\tilde{\pi}^{(2)}$, by contrast with the free-surface source density $\pi^{(2)}$ in (3.27), depend explicitly upon the first-order mapping $\boldsymbol{\xi}^{(1)}$. This fact is exploited below, where $\tilde{\pi}^{(2)}$ and $\tilde{\tau}^{(2)}$ are successively cancelled by properly selecting the mapping $\boldsymbol{\xi}^{(1)}$ and the 'linearized hull' $f^{(1)}$. It turns out that the free-surface source density $\tilde{\pi}^{(2)}$ is identically cancelled by Guilloton's mapping, which is thus given a precise interpretation in the frame of the present analysis. By generalizing Guilloton's mapping, the volume distribution of sources $\delta^{(2)}$ can also be cancelled on the centre-plane (not for $|z| > 0$, however).

Before we proceed with the successive cancellation of $\tilde{\pi}^{(2)}$, $\tilde{\tau}^{(2)}$ and (on $z = 0$) $\delta^{(2)}$, it may be worth emphasizing the purpose of the operation and the role of the mapping $\boldsymbol{\xi}^{(1)}$. By (3.31 *a*, *b*) and (5.6) we have

$$\left.\begin{aligned} U_i(\mathbf{X}) &= \epsilon u_i^{(1)}(\mathbf{x}) - \epsilon^2 u_j^{(1)}\xi_{j,i}^{(1)} + \epsilon^2\theta_{,i}^{(2)} + O(\epsilon^3) \quad (i = 1,2,3), \\ \mathbf{X} &= \mathbf{x} + \epsilon\boldsymbol{\xi}^{(1)}(\mathbf{x}) + O(\epsilon^2). \end{aligned}\right\} \tag{5.11}$$

It is again emphasized that $\boldsymbol{\xi}^{(1)}$ has no effect upon $\mathbf{U}(\mathbf{X})$ to order ϵ^2. However, our aim is to cancel $\theta^{(2)}$, or at least render it negligibly small, by selecting a suitable first-order mapping vector $\boldsymbol{\xi}^{(1)}$ and 'linearized hull' $f^{(1)}$. Thus we seek to incorporate the nonlinear effects associated with the second-order potential $\theta^{(2)}$ into the first-order velocity $\mathbf{u}^{(1)}$ [by means of the 'linearized hull' $f^{(1)}$, see (3.16)] and the second-order term $u_j^{(1)}\xi_{j,i}^{(1)}$ (this term can be easily evaluated in terms of $\mathbf{u}^{(1)}$ and $\boldsymbol{\xi}^{(1)}$).

By intuitive physical reasoning, Guilloton (1964) arrived at the transformation† (interpreted here as a first-order mapping‡ $\boldsymbol{\xi}^{(1)}$)

$$\left.\begin{aligned} \xi^{(1)}(\mathbf{x}) &= \int_{-\infty}^{x} u^{(1)}(x',y,z)\,dx', \quad \eta^{(1)}(\mathbf{x}) = -u^{(1)}(\mathbf{x}), \\ \zeta^{(1)}(\mathbf{x}) &= \int_{-\infty}^{x} w^{(1)}(x',y,z)\,dx'. \end{aligned}\right\} \tag{5.12}$$

† In fact, the mapping (5.12) differs slightly from Guilloton's transformation by a second-order term in $\xi^{(1)}$, extension of the integration in x' to infinity upstream (instead of the bow) and extension of the mapping to the entire space; for details see Dagan (1975*a*) and Noblesse (1975*a*).

‡ It may be readily verified that (3.8) and (3.11) are satisfied.

We may immediately verify that Guilloton's mapping (5.12) has the distinguished property of rendering the strength $\tilde{\pi}^{(2)}(x,z)$ of the free-surface sources (5.9) equal to zero. Guilloton's mapping (5.12), however, does not cancel the centre-plane and volume sources $\tilde{\tau}^{(2)}$ and $\delta^{(2)}$. Indeed, by substituting (5.12) into (5.8) and using (3.7 a,b) and (3.9), we obtain

$$\tilde{\tau}^{(2)}(x,y) = f_x^{(2)} - f_x^{(1)}\int_{-\infty}^{x} v_y^{(1)}\,dx' - v^{(1)}f_{xx}^{(1)} + (u_x^{(1)}+v^{(1)})f_y^{(1)} \quad (z=0). \tag{5.13}$$

The method of Guilloton implies that $\tilde{\tau}^{(2)}$ and $\delta^{(2)}$ may be neglected. However, it will now be shown that $\tilde{\tau}^{(2)}$ may easily be cancelled by extending Guilloton's concept of the 'linearized hull'. Guilloton's 'linearized hull' function $f^{(1)}$ is defined by the mapping (5.12) and the hull expansion (3.5), where $f^{(2)}$ is set equal to zero. It is clear, however, that $f^{(2)}$ need not be set equal to zero, and (5.13) then shows that the second-order centre-plane sources $\tilde{\tau}^{(2)}$ may be rendered equal to zero by taking

$$f^{(2)}(x,y) = \int_{-\infty}^{x}\left[f_x^{(1)}(x',y)\int_{-\infty}^{x} v_y^{(1)}(x'',y,0)\,dx'' + v^{(1)}f_{xx}^{(1)} - (u_x^{(1)}+v^{(1)})f_y^{(1)}\right]dx' \quad (z=0). \tag{5.14}$$

Equations (3.5) and (5.12) then give

$$f^{(1)}(x,y) = F^{(1)}\left\{x+\epsilon\int_{-\infty}^{x} u^{(1)}(x',y,0)\,dx', y-\epsilon u^{(1)}(x,y,0)\right\} - \epsilon f^{(2)}(x,y). \tag{5.15}$$

The function $f^{(1)}$ enters implicitly on the right-hand side of (5.15) through $u^{(1)}$ [see (3.12) and (3.16)] and $f^{(2)}$ [see (5.14)], and must then be found by means of an iterative procedure.

Thus Guilloton's mapping (5.12) renders the second-order free-surface sources $\tilde{\pi}^{(2)}$ equal to zero, and the generalized 'linearized hull' $f^{(1)}$ in (5.15) makes the second-order centre-plane sources $\tilde{\tau}^{(2)}$ zero. It now follows from (5.10) and (5.11) that the second-order approximation to the velocity is given by

$$\left.\begin{aligned} U_i(\mathbf{X}) &= \epsilon u_i^{(1)}(\mathbf{x}) - \epsilon^2 u_j^{(1)}\xi_{j,i}^{(1)} + \frac{\epsilon^2}{4\pi}\iiint G_{,i}(\mathbf{x};\mathbf{x}')\,\delta^{(2)}(\mathbf{x}')\,d\mathbf{x}' \quad (i=1,2,3),\\ \mathbf{X} &= \mathbf{x}+\epsilon\boldsymbol{\xi}^{(1)}(\mathbf{x}). \end{aligned}\right\} \tag{5.16}$$

If the volume distribution of sources is neglected, we obtain an inconsistent second-order solution which satisfies the irrotationality equations (3.18), the hull condition (3.20) and the free-surface condition (3.23), but not the continuity equation (3.17). The main feature of this inconsistent approximation is that it is directly given in terms of the first-order solution $\mathbf{u}^{(1)}$ by means of relatively simple expressions.

This approximation differs from that of Guilloton by the term $u_j^{(1)}\xi_{j,i}^{(1)}$ in (5.16) [Guilloton's approximation is of the form (5.1 a,b), i.e. $\mathbf{u}^{(2)}$ is zero] and by the second-order term $f^{(2)}$ in (5.15). Similar inconsistent approximations were also previously obtained by Wehausen (1969), Dagan (1975 a) and Noblesse (1975 a). These approximations are based on the Lagrangian mapping

$$\boldsymbol{\xi}^{(1)}(\mathbf{x}) = \int_{-\infty}^{x} \mathbf{u}^{(1)}(x',y,z)\,dx',$$

which differs slightly from Guilloton's mapping (5.12) in the vertical component $\eta^{(1)}$. The main differences between the above inconsistent approximation and those of Dagan (1975*a*) and Noblesse (1975*a*) lie in the term $f^{(2)}$, which was taken equal to zero, and in the term $u_j^{(1)}\xi_{j,i}^{(1)}$ in (5.16), in place of which we had the second-order velocity† $u^{(2)} = -\frac{1}{2}\mathbf{u}^{(1)}.\mathbf{u}^{(1)}$, $v^{(2)} = w^{(2)} = 0$.

The above comparison between the various approximate solutions is of course not intended to suggest that the particular approximation obtained above should be expected to lead to numerical results in better agreement with experimental measurements. All these various approximations are inconsistent second-order solutions, and there is clearly no *a priori* reason why neglecting to satisfy (to second order) the continuity equation only, rather than, say, both the equation of continuity and the equations of irrotationality as in the previous studies of Dagan (1975*a*) and Noblesse (1975*a*) should lead to a better approximation.

In any case, we shall now indicate how the volume distribution of sources $\delta^{(2)}$ can be cancelled on the centre-plane by generalizing Guilloton's mapping (5.12) [one must hope that this would render $\delta^{(2)}$ small in the neighbourhood of the hull, and the last term in (5.16) negligible]. It will readily be verified that the free-surface sources $\bar{\pi}^{(2)}$ [see (5.9)] are identically cancelled by the mapping

$$\xi^{(1)} = \int_{-\infty}^{x} u^{(1)}\,dx' + \lambda^{(1)}, \quad \eta^{(1)} = -u^{(1)} + \mu^{(1)}, \quad \zeta^{(1)} = \int_{-\infty}^{x} w^{(1)}\,dx', \tag{5.17}$$

where the functions $\lambda^{(1)}$ and $\mu^{(1)}$ satisfy the conditions

$$\lambda_{xx}^{(1)} + \lambda_{y}^{(1)} = 0, \quad \mu^{(1)} = 0, \quad \mu_y^{(1)} = 2\lambda_x^{(1)} \quad (y = 0). \tag{5.18}$$

In addition, we may require $\lambda^{(1)}$ to satisfy the Laplace equation $\nabla^2\lambda^{(1)} = 0$, so that $\lambda^{(1)}$ may be written as

$$\lambda^{(1)}(\mathbf{x}) = \frac{1}{2\pi}\iint_{\sigma} G(\mathbf{x}; x', y', 0)\,\tau^{(1)}(x', y')\,dx'\,dy', \tag{5.19}$$

where $\tau^{(1)}(x, y)$ is related to $\lambda^{(1)}$ by $\lambda_z^{(1)}(x, y, \pm 0) = \pm\tau^{(1)}(x, y)$. Equation (5.18) may be satisfied in a simple way by taking, for instance,

$$\mu^{(1)}(\mathbf{x}) = 2\lambda_x^{(1)}(x, 0, z)\,\beta(y), \tag{5.20}$$

where $\beta(y)$ is a function satisfying $\beta(0) = 0$, $\beta'(0) = 1$ and $\beta \to 0$ as $y \to -\infty$. It may be noted that (5.20) implies that $\mu^{(1)}$ satisfies a Poisson equation rather than Laplace's equation.

Substituting (5.17), (5.19) and (5.20) into (5.7) and requiring that $\delta^{(2)} = 0$ on $z = 0$ leads to a two-dimensional inhomogeneous integral equation of the second kind for the centre-plane source density $\tau^{(1)}(x, y)$ in (5.19). This integral equation may be simplified to a one-dimensional equation if $\delta^{(2)}$ is required to vanish only along the centre-line $y = z = 0$ rather than on the centre-plane $z = 0$.

Admittedly, the mapping (5.17) can hardly be regarded as a simple mapping in comparison with Guilloton's mapping (5.12). However, it must be remembered

† The term $u^{(2)} = -\frac{1}{2}\mathbf{u}^{(1)}.\mathbf{u}^{(1)}$ was found necessary to cancel the free-surface sources. In the present approximate solution, we have $u^{(2)} = -\mathbf{u}^{(1)}.\mathbf{u}^{(1)}$.

that we are seeking a mapping which accounts for the effects of the second-order potential $\theta^{(2)}$, and perhaps one should not expect too simple an answer. The difficulties of a solution along the lines described above must really be weighed against the difficulties associated with the evaluation of the contribution of the free-surface sources $\pi^{(2)}$ in (3.27) to $\phi^{(2)}$ in (3.28). A definite advantage of the solution presented above is that it involves centre-plane source distributions alone. This means that the Havelock source potential $G(\mathbf{x};\mathbf{x}')$ need be considered only for $z = z' = 0$, so that the double and simple integrals in $G(\mathbf{x};\mathbf{x}')$, which are functions of the three variables $x-x'$, $y+y'$ and $z-z'$, become functions of two variables only.

6. Summary and concluding remarks

In this paper, the classical thin-ship theory and Guilloton's method have been derived by a rational and unified approach based on a continuous mapping of the flow domain onto a simple reference domain and perturbation expansions for the velocity and mapping vectors with the beam/length ratio as the small parameter. The classical thin-ship theory was obtained in § 4 as the consistent perturbation solution in physical space. This consistent solution is independent of the mapping and is expressed explicitly in terms of the hull function.

In contrast, the perturbation solution in § 5.2 is expressed in parametric form, with the reference-space co-ordinates appearing as parameters. Such a parametric representation, unlike the explicit representation of the classical thin-ship theory, is not unique, as testified by the fact that the mapping of the flow domain onto the reference domain is arbitrary to a certain extent. This arbitrariness was exploited to derive a second-order approximation expressed in terms of the first-order velocity field generated by the 'linearized hull'. The latter is related to the real hull by means of Guilloton's mapping (5.12) or the generalized mapping (5.17). Guilloton's mapping has been shown to cancel the free-surface sources $\tilde{\pi}^{(2)}$ in (5.9), while the mapping (5.17) aims at cancelling both $\tilde{\pi}^{(2)}$ and the volume sources $\delta^{(2)}$ in (5.7) along the centre-plane. In physical terms, this means that part of the nonlinear second-order free-surface effects are accounted for by the first-order velocity field (Michell's linear solution) generated by Guilloton's 'linearized hull', while the generalized 'linearized hull' based on (5.17) would incorporate the nonlinear free-surface effects entirely.

An interesting feature of the parametric representations of the second-order approximation, either in the form (3.31) or (5.11), is that the perturbation parameter ϵ enters implicitly and therefore to order higher than the second. In other words, such a second-order approximation actually includes higher-order terms, so that it is not strictly consistent to second order. This may be illustrated by considering the wave profile along the hull, which for simplicity may be written as $Y = \epsilon a \sin(X-\epsilon\xi^{(1)})$, where a is some slowly varying amplitude function. If the bow (or the stern) of the ship is fairly blunt, $\xi^{(1)}$ may become large there, so that expanding $\sin(X-\epsilon\xi^{(1)})$ asymptotically as $\sin X - \epsilon\xi^{(1)}\cos X + O(\epsilon^2)$ would result in poor accuracy. In fact, if the bow is too blunt, such an asymptotic expansion would become non-uniform. Similar difficulties occur at low Froude number, as noted in §4. These difficulties, however, do not occur if the solution is expressed in implicit form, i.e. as $Y = \epsilon a \sin x$, $X = x + \epsilon\xi^{(1)}$.

The intention of the above discussion was to point out that it may be advantageous to express perturbation solutions in parametric form. It is not claimed however that the second-order approximation (5.16) is free from non-uniformities at the bow and stern and at low Froude number, since there are second-order terms in the expansion for the velocity which may lead to non-uniformities. In fact, we have been concerned in this paper exclusively with the regular perturbation problem, ignoring the question of the uniformity or non-uniformity of the perturbation solution. Yet it is tempting to speculate that the generalized mapping (5.17) might render the second-order solution (5.16) uniform by cancelling the volume sources $\delta^{(2)}$ on the centre-plane, and in particular at the bow and stern, where non-uniformities originate.

There is obviously a certain similarity between the continuous-mapping approach and the method of co-ordinate straining; indeed the reference-space co-ordinates $\mathbf{x}$ and the mapping vectors $\boldsymbol{\xi}^{(n)}$ can be regarded as strained co-ordinates and straining functions, respectively. Both the mapping and the straining of co-ordinates are non-unique, and the perturbation solution is usually expressed in parametric form. There is, however, an important difference between the two approaches: co-ordinate straining aims at eliminating non-uniformities from perturbation solutions (Van Dyke 1975, p. 99), whereas the mapping is used here strictly in the context of a regular perturbation analysis, as noted above. In addition, the method of co-ordinate straining has been applied mainly to two-dimensional flow problems whereas here we are concerned with a complex three-dimensional flow with a nonlinear free-surface boundary condition.

The nonlinear approximation presented in §5.2 must ultimately be validated by comparing the predicted wave resistance and wave pattern with measurements for hulls of practical shapes. This requires elaborate numerical computations, which are at present being undertaken by the first author, who is mainly responsible for the developments of §5.2. Other applications and extensions of the approach developed in this paper are also contemplated by the authors.

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Appendix A. Solution of the field equations by Taylor expansions

By (2.5) we have $\mathbf{U}(\mathbf{X}) = \mathbf{U}(\mathbf{x}+\boldsymbol{\xi})$. Expanding in a Taylor series yields

$$\mathbf{U}(\mathbf{X}) = \mathbf{U}(\mathbf{x}) + (\boldsymbol{\xi}\,.\,\nabla)\,\mathbf{U}(\mathbf{x}) + \tfrac{1}{2}(\boldsymbol{\xi}\,.\,\nabla)^2\,\mathbf{U}(\mathbf{x}) + \dots. \qquad \text{(A 1)}$$

We assume that $\mathbf{U}(\mathbf{X})$ has the asymptotic expansion

$$\mathbf{U}(\mathbf{X}) = \epsilon\mathbf{U}^{(1)}(\mathbf{X}) + \epsilon^2\mathbf{U}^{(2)}(\mathbf{X}) + \dots \qquad \text{(A 2)}$$

in the physical space $\mathbf{X}$. The field equations (2.1) imply that

$$\nabla\,.\,\mathbf{U}^{(n)}(\mathbf{X}) = \nabla \times \mathbf{U}^{(n)}(\mathbf{X}) = 0 \quad (n \geqslant 1).$$

Therefore we have

$$\mathbf{U}^{(n)}(\mathbf{X}) = \nabla\phi^{(n)}(\mathbf{X}), \quad \nabla^2\phi^{(n)} = 0 \quad (n \geqslant 1).$$

Expansion (A 2) then becomes

$$\mathbf{U}(\mathbf{X}) = \epsilon\nabla\phi^{(1)}(\mathbf{X}) + \epsilon^2\nabla\phi^{(2)}(\mathbf{X}) + \dots. \tag{A 3}$$

This gives

$$\mathbf{U}(\mathbf{x}) = \epsilon\nabla\phi^{(1)}(\mathbf{x}) + \epsilon^2\nabla\phi^{(2)}(\mathbf{x}) + \dots. \tag{A 4}$$

Finally, by substituting (A 4) and (3.4) into (A 1) and using (2.6) and (3.3), we obtain (3.12) and (3.24).

More generally, it follows from the above analysis that the general solution of the field equations (2.12) in the reference space is given by

$$\mathbf{u}(\mathbf{x}) = [1 + \boldsymbol{\xi}.\nabla + \tfrac{1}{2}(\boldsymbol{\xi}.\nabla)^2 + \dots]\,\nabla\phi(\mathbf{x}), \tag{A 5}$$

where ϕ satisfies the Laplace equation $\nabla^2\phi = 0$.

The above approach follows Joseph (1973).

Appendix B. Hydrodynamic forces and moment

A general expression for the wave resistance experienced by the ship may be obtained from considerations of momentum balance; see Wehausen (1973, p. 100). For an inviscid fluid, we have

$$R = \int_{S_c} [-U(\mathbf{U}.\mathbf{n}) + \tfrac{1}{2}(U^2 + V^2 + W^2)\,n_x]\,dS - \frac{1}{2}\oint_{C_c} E^2\,dZ, \tag{B 1}$$

where $R = R'g^2/\rho\mathscr{U}^6$, S_c is an arbitrary control surface drawn in the fluid and surrounding the hull, C_c denotes the line where S_c intersects the free surface, and $\mathbf{n}$ is a unit vector normal to S_c and pointing outwards.

The control surface S_c may be taken as the wetted hull, in which case we have $\mathbf{U}.\mathbf{n} = -n_x$. By transforming the surface integral over the wetted hull into a double integral over the projection Σ_{wh} of the wetted hull onto the centre-plane, and by a similar transformation of the line integral, we then obtain

$$R = -2\iint_{\Sigma_{wh}} [U + \tfrac{1}{2}(U^2 + V^2 + W^2)]\,F_X\,dX\,dY - \int_{X_B}^{X_S} E^2\frac{F_X + F_Y E_X}{1 - F_Y E_Z}\,dX, \tag{B 2}$$

where X_B and X_S denote the abscissae of the bow and stern of the ship. An equivalent expression may be obtained by integrating the pressure acting upon the hull.

By using the mapping relations (2.5)–(2.9), the wave resistance R given by (B 2) may then be expressed in terms of quantities in the reference space:

$$R = -2\iint_{\sigma} [u + \tfrac{1}{2}(u^2 + v^2 + w^2)]\,(f_x + f_x\eta_y - f_y\eta_x)\,dx\,dy - \int_{x_B}^{x_S} \eta^2 f_x\,dx. \tag{B 3}$$

Finally, introducing the perturbation expansions (3.2)–(3.4) yields, after an easy manipulation cancelling the simple integral in (B 3),

$$R = -2\iint_\sigma \epsilon u^{(1)}(\epsilon f_x^{(1)} + \epsilon^2 f_x^{(2)})\,dx\,dy$$
$$-2\epsilon^3 \iint_\sigma [u^{(2)} + \tfrac{1}{2}(u^{(1)2} + v^{(1)2} + w^{(1)2})] f_x^{(1)} dx\,dy$$
$$-2\epsilon^3 \iint_\sigma (u^{(1)} + \eta^{(1)})\,(f_x^{(1)}\eta_y^{(1)} - f_y^{(1)}\eta_x^{(1)})\,dx\,dy + O(\epsilon^4). \qquad \text{(B 4)}$$

By (3.11), we have $u^{(1)} + \eta^{(1)} = 0$ on $y = 0$, so that the last term in (B 4) is actually of order ϵ^4 for ships whose draft is as small as the beam. In this case, (B 4) may be written as

$$R = -2\iint_\sigma [\epsilon u^{(1)} + \tfrac{1}{2}\epsilon^2(u^{(1)2} + v^{(1)2} + w^{(1)2}) + \epsilon^2 u^{(2)}] f_x\,dx\,dy + O(\epsilon^4), \qquad \text{(B 5)}$$

where expansion (3.2) has been used.

An expression for the wave resistance in terms of the far-field free-waves may also be derived by choosing the control surface S_c in (B 1) as a vertical plane far upstream from the ship together with a surface S_∞ far downstream extending to infinity downwards and in the lateral direction. The corresponding expression is directly given by (B 1) with S_c and C_c replaced by S_∞ and C_∞ (the line where S_∞ intersects the free surface).

It is convenient to take S_∞ as the surface which is mapped onto a vertical plane $x = x_\infty$, $y \leqslant 0$ in the reference domain. By using the relationship

$$\mathbf{n}\,dS = (\nabla y + \boldsymbol{\xi}_y) \times (\nabla z + \boldsymbol{\xi}_z)\,dy\,dz,$$

(B 1) becomes

$$R = \int_{-\infty}^{0} dy \int_0^\infty dz [(-u^2 + v^2 + w^2)(1 + \eta_y + \zeta_z) + 2u(v\xi_y + w\xi_z)$$
$$+ (u^2 + v^2 + w^2)(\eta_y\zeta_z - \eta_z\zeta_y) - 2u\mathbf{u}\,.\,(\boldsymbol{\xi}_y \times \boldsymbol{\xi}_z)]_{x=x_\infty}$$
$$+ \int_0^\infty \eta^2(x_\infty, 0, z)\,[1 + \zeta_z(x_\infty, 0, z)]\,dz. \qquad \text{(B 6)}$$

By substituting the asymptotic expansions (3.3) and (3.4) and by using (3.11) and (3.22), (B 6) becomes

$$R = \epsilon^2 \left\{ \int_{-\infty}^{0} \int_0^\infty (-u^{(1)2} + v^{(1)2} + w^{(1)2})\,dy\,dz + \int_0^\infty u^{(1)2}\,dz \right\}$$
$$+ \epsilon^3 \left\{ 2\int_{-\infty}^{0} dy \int_0^\infty dz [-u^{(1)}u^{(2)} + v^{(1)}v^{(2)} + w^{(1)}w^{(2)} \right.$$
$$+ \tfrac{1}{2}(-u^{(1)2} + v^{(1)2} + w^{(1)2})(\eta_y^{(1)} + \zeta_z^{(1)}) + u^{(1)}(v^{(1)}\xi_y^{(1)} + w^{(1)}\xi_z^{(1)})]$$
$$\left. + \int_0^\infty [2u^{(1)}u^{(2)} + u^{(1)}(u^{(1)2} + v^{(1)2} + w^{(1)2}) + u^{(1)2}\zeta_z^{(1)}]\,dz \right\}$$
$$+ O(\epsilon^4). \qquad \text{(B 7)}$$

As $x_\infty \to \infty$, $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ tend to zero like $x^{-\frac{1}{2}}$. If $\boldsymbol{\xi}^{(1)}$ is also assumed to vanish like $x^{-\frac{1}{2}}$ (as is the case for both Guilloton's mapping and the suggested generalized Guilloton mapping), it may be seen that the last two terms in both the double and the simple integral in the bracket $O(\epsilon^3)$ vanish like $x^{-\frac{3}{2}}$ and may thus be dropped. Expansion (B 7) may then be written as

$$R = \int_{-\infty}^{0}\int_{0}^{\infty}[-(\epsilon u^{(1)}+\epsilon^2 u^{(2)})^2+(\epsilon v^{(1)}+\epsilon^2 v^{(2)})^2+(\epsilon w^{(1)}+\epsilon^2 w^{(2)})^2]\,dy\,dz + \int_{0}^{\infty}(\epsilon u^{(1)}+\epsilon^2 u^{(2)})^2\,dz+O(\epsilon^4). \tag{B 8}$$

In addition to the wave resistance R, the ship also experiences a vertical lift $L = L'g^2/\rho\mathscr{U}^6$ (counted as positive if acting upwards) and a moment $M = M'g^3/\rho\mathscr{U}^8$ about the Z axis (evaluated at the origin O of the system of co-ordinates $OXYZ$ and counted as positive if acting anticlockwise). These may readily be obtained by integrating the pressure acting upon the hull:

$$L = -2\iint_{\Sigma_{wh}}[U+\tfrac{1}{2}(U^2+V^2+W^2)+Y]\,F_Y\,dX\,dY, \tag{B 9}$$

$$M = -2\iint_{\Sigma_{wh}}[U+\tfrac{1}{2}(U^2+V^2+W^2)+Y]\,(XF_Y-YF_X)\,dX\,dY. \tag{B 10}$$

A simple manipulation of the hydrostatic terms yields

$$L = 2\iint_{\Sigma}F\,dX\,dY+L_*, \quad M = 2\iint_{\Sigma}FX\,dX\,dY+M_*, \tag{B 11 a, b}$$

with L_* and M_* defined as

$$L_* = -2\iint_{\Sigma_{wh}}[U+\tfrac{1}{2}(U^2+V^2+W^2)]\,F_Y\,dX\,dY + 2\int_{X_B}^{X_S}\left\{\int_0^E F(X,Y)\,dY - F(X,E)\,E\right\}dX, \tag{B 12}$$

$$M_* = -2\iint_{\Sigma_{wh}}[U+\tfrac{1}{2}(U^2+V^2+W^2)]\,(XF_Y-YF_X)\,dX\,dY + 2\int_{X_B}^{X_S}\left\{\int_0^E F(X,Y)\,dY - F(X,E)\,E\right\}X\,dX. \tag{B 13}$$

In (B 11 *a, b*), Σ denotes the centre-plane area inside the hull and below $Y = 0$ (Σ must not be confused with Σ_{wh}, which is the centre-plane area inside the hull and below the free-surface profile $Y = E$). The first terms in (B 11 *a, b*) depend on the hull geometry alone, and can thus exactly, and easily, be calculated. Indeed, the term $2\iint_{\Sigma}F\,dX\,dY$ is nothing but the volume enclosed by the hull below $Y = 0$, and corresponds to the buoyancy force for this volume. The term $2\iint_{\Sigma}FX\,dX\,dY$ similarly corresponds to the moment of this hydrostatic force.

The last terms in (B 12) and (B 13), on the other hand, are readily seen to be identically zero for a hull with vertical sides in the neighbourhood of the free surface, and negligible for most hulls of practical shape. In any case, in a strict asymptotic sense, these terms are of order ϵ^3. The first terms in (12) and (13) may be expressed in terms of quantities in the reference space in the manner shown for the wave resistance. Introducing expansions (3.2)–(3.4) then gives, to second order,

$$L_* = -2\epsilon^2 \iint_\sigma u^{(1)} f_y^{(1)}\, dx\, dy + O(\epsilon^3), \tag{B 14}$$

$$M_* = -2\epsilon^2 \iint_\sigma u^{(1)} (x f_y^{(1)} - y f_x^{(1)})\, dx\, dy + O(\epsilon^3). \tag{B 15}$$

For the case of ships whose draft is as small as the beam, y in (B 15) is of order ϵ and the corresponding term then becomes of order ϵ^3. Furthermore, (B 14) and (B 15) may be simplified by neglecting the variation of $u^{(1)}$ (of order ϵ) over the draft (it must nevertheless be remembered that the wave component of $u^{(1)}$ dies out exponentially, i.e. fast). We then readily obtain

$$L_* = -2\epsilon^2 \int_{x_B}^{x_S} u^{(1)} f^{(1)}\, dx + O(\epsilon^3), \quad M_* = -2\epsilon^2 \int_{x_B}^{x_S} u^{(1)} f^{(1)} x\, dx + O(\epsilon^3). \tag{B 16}$$

Equations (B 14)–(B 16) are in agreement with the results obtained by Wehausen (1969) in his study of the use of Lagrangian co-ordinates for ship waves.

Appendix C. Sinkage and trim

Let $(\tilde{X}, \tilde{Y})$ be a system of co-ordinates attached to the hull and such that $\tilde{X} \equiv X$ and $\tilde{Y} \equiv Y$ when the ship is at rest. The ship in motion experiences a certain lift L, wave resistance R and moment M, and, in response to these, a certain sinkage h (defined as the downward vertical displacement at the origin O of the co-ordinate system $OXYZ$) and trim angle α (counted as positive for a bow-up rotation); see figure 2. The relationship between $(\tilde{X}, \tilde{Y})$ and (X, Y) is

$$\tilde{X} = X \cos\alpha - (Y + h) \sin\alpha, \quad \tilde{Y} = X \sin\alpha + (Y + h) \cos\alpha. \tag{C 1}$$

The unknown hull equation $Z = \pm F(X, Y)$ is then related to the given equation $Z = \pm \tilde{F}(\tilde{X}, \tilde{Y})$ by means of (C 1) together with the relation $F(X, Y) = \tilde{F}(\tilde{X}, \tilde{Y})$. By using (2.9), (2.8) and (C 1), we then obtain

$$\begin{aligned} f(x,y) = \tilde{F}\{&[x + \xi(x,y,0)] \cos\alpha - [y + \eta(x,y,0) + h] \sin\alpha, \\ &[x + \xi(x,y,0)] \sin\alpha + [y + \eta(x,y,0) + h] \cos\alpha\}, \end{aligned} \tag{C 2}$$

which relates the 'linearized hull' function $f(x,y)$ to the given hull function $\tilde{F}(\tilde{X}, \tilde{Y})$. The effects of sinkage and trim are thus indirectly incorporated into the solution by means of the 'linearized hull' function $f(x,y)$.

The sinkage h and the trim α are determined by the condition that the ship, regarded as a free body, is in equilibrium under the action of the various forces acting upon it. The equations stating the equilibrium of the ship are

$$R - T\cos\alpha = 0, \tag{C 3}$$

$$L - W + T\sin\alpha = 0, \tag{C 4}$$

$$M - M_w - T(h\cos\alpha + d_T) = 0, \tag{C 5}$$

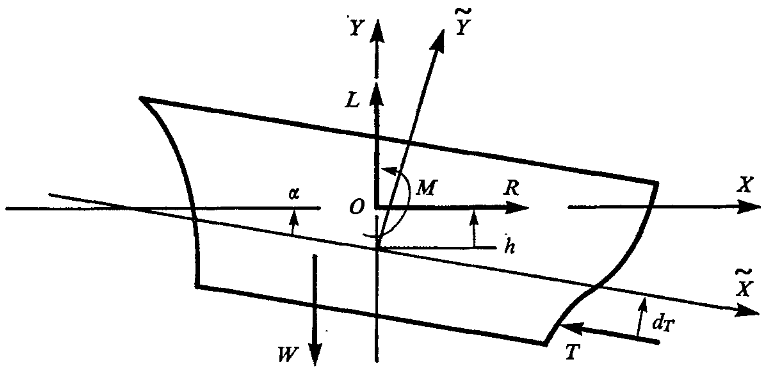

FIGURE 2. Definition sketch for sinkage and trim.

where R, L and M denote the wave resistance, the lift and the moment exerted upon the ship by the water, W and M_w denote the weight of the ship and its moment with respect to the origin O, and T is the propelling thrust, assumed to be acting parallel to the $\tilde{X}$ axis at a distance d_T below it; see figure 2. The condition for equilibrium of the ship at rest implies

$$W = 2\iint_{\Sigma_{eq}} F\,dX\,dY, \quad M_w = 2\iint_{\Sigma_{eq}} FX\,dX\,dY, \tag{C 6}$$

where Σ_{eq} denotes the projection of the wetted hull of the ship at rest onto the centre-plane. By replacing T by $R\sec\alpha$ in (C 4) and (C 5), and by using (C 6) and (B 11 a, b), we then obtain

$$2\iint_{\Sigma} F\,dX\,dY - 2\iint_{\Sigma_{eq}} F\,dX\,dY + L_* + R\tan\alpha = 0, \tag{C 7}$$

$$2\iint_{\Sigma} FX\,dX\,dY - 2\iint_{\Sigma_{eq}} FX\,dX\,dY + M_* - R(h + d_T\sec\alpha) = 0. \tag{C 8}$$

By noting that a downward displacement h (sinkage) and a bow-up rotation α (trim) of the ship is equivalent to an upward elevation h and opposite rotation α of the free surface, we easily see that

$$\iint_{\Sigma} F(X,Y)\,dX\,dY - \iint_{\Sigma_{eq}} F(X,Y)\,dX\,dY = \int dX \int_0^{h+X\tan\alpha} F(X,Y)\,dY$$
$$= \int_{X_B}^{X_S} (h + X\tan\alpha)\,F(X,0)\,dX + O(\epsilon^2) \tag{C 9}$$

for h and α small, of order ϵ (a will be shown below to be the case), and similarly for FX in (C 8). With h and α assumed to be of order ϵ, as well as d_T, the terms involving the wave resistance R, itself of order ϵ^2, in (C 7) and (C 8) are seen to be of order ϵ^3. Thus, to order ϵ^2, (C 7)–(C 9) give

$$h\int_{X_B}^{X_S} B(X)\,dX + \alpha\int_{X_B}^{X_S} B(X)\,X\,dX + \tfrac{1}{2}L_* = 0, \tag{C 10}$$

$$h\int_{X_B}^{X_S} B(X)\,X\,dX + \alpha\int_{X_B}^{X_S} B(X)\,X^2\,dX + \tfrac{1}{2}M_* = 0, \tag{C 11}$$

where $B(X) = F(X,0)$ denotes the half-beam of the ship at section X. Equations (C 10) and (C 11) are a system of linear algebraic equations for the sinkage h and trim α. The coefficients of h and α depend on the geometry of the hull water-plane section alone. Since these coefficients are of order ϵ while L_* and M_* are $O(\epsilon^2)$, it follows that h and α are $O(\epsilon)$, as assumed above. Equations (C 10) and (C 11) have been given by Tuck (1966) and, with L_* and M_* given by (B 16), by Wehausen (1969). The approach used above is different from that of Wehausen, who introduced sinkage and trim explicitly into the analysis from the outset.

In the particular case of a fixed hull ($h = \alpha = 0$), the 'linearized hull' function $f(x,y)$ must be determined from (C 2) by an iterative process, as the mapping components ξ and η depend on the velocity field, which itself depends on f. A convenient first approximation is obviously $f(x,y) = \tilde{F}(x,y)$; see for instance Gadd (1973). If sinkage and trim are allowed, this iterative procedure need be modified only by using (C 10) and (C 11) with L_* and M_* given by (B 16).

Finally, it should be recalled that the expressions used for R, L_* and M_* in (C 7) and (C 8) are based on the assumption of inviscid fluid. Viscosity effects, of course, result in changes in R and, to a lesser degree, in L_* and M_*. It may readily be seen, however, that expressions (C 10) and (C 11) for h and α remain valid if the changes in R, on the one hand, and in L_* and M_*, on the other hand, are of order ϵ^2 and ϵ^3, respectively. This is not verified at low speed, when the resistance experienced by the ship is mainly due to viscous effects, but then sinkage and trim are negligible. In fact, Yeung (1972) computed sinkage and trim by using (C 10) and (C 11) with L_* and M_* given by (B 16), and obtained satisfactory agreement with experimental values [although (C 10) and (C 11) are not valid at low Froude number F, they predict $h \to 0$ and $\alpha \to 0$ as $F \to 0$ since L_*, $M_* \to 0$ as $F \to 0$].

REFERENCES

DAGAN, G. 1973 Waves and wave resistance of thin bodies moving at low speed. *Hydronautics Inc. Tech. Rep.* no. 7103-5.

DAGAN, G. 1975*a* A method of computing nonlinear wave resistance of thin ships by coordinate straining. *J. Ship Res.* **19**, 149–154.

DAGAN, G. 1975*b* Waves and wave resistance of thin bodies moving at low speed: the free-surface nonlinear effect. *J. Fluid Mech.* **69**, 405–417.

EGGERS, K. W. H. 1966 Second-order contributions to ship waves and wave resistance. *Proc. 6th Symp. Naval Hydrodyn.*, pp. 649–672.

GADD, G. E. 1973 Wave resistance calculations by Guilloton's method. *Trans. Roy. Inst. Naval Archit.* **115**, 377–384.

GUILLOTON, R. 1964 L'étude théorique du bateau en fluide parfait. *Bull. Assoc. Tech. Maritime Aero.* **64**, 537–552.

JOSEPH, D. D. 1973 Domain perturbations: the higher order theory of infinitesimal water waves. *Arch. Rat. Mech. Anal.* **51**, 297–302.

LANDWEBER, L. 1973 Contribution on some current problems of ship resistance. *Proc. Int. Jubilee Meeting Netherlands Ship Model Basin*, pp. 32–46.

MARUO, H. 1966 A note on the higher-order theory of thin ships. *Bull. Fac. Engng Yokohama Nat. Univ.* **15**, 1–21.

MICHELL, J. H. 1898 The wave resistance of a ship. *Phil. Mag.* **45**, 106–123.

NOBLESSE, F. 1975*a* A perturbation analysis of the wavemaking of a ship, with an interpretation of Guilloton's method. *J. Ship Res.* **19**, 140–148.

NOBLESSE, F. 1975*b* The wavemaking of a ship: approximate solutions based on a parametric formulation. *Iowa Inst. Hydr. Res. Rep.* no. 172.

PETERS, A. S. & STOKER, J. J. 1957 The motion of a ship, as a floating rigid body, in a seaway. *Comm. Pure Appl. Math.* **10**, 399–490.

SALVESEN, N. 1969 On higher-order wave theory for submerged two-dimensional bodies. *J. Fluid Mech.* **38**, 415–432.

TUCK, E. O. 1966 Shallow-water flow past slender bodies. *J. Fluid Mech.* **26**, 81–95.

VAN DYKE, M. 1975 *Perturbation Methods in Fluid Mechanics.* Stanford, California: Parabolic Press.

WEHAUSEN, J. V. 1963 An approach to thin-ship theory. *Int. Sem. Theor. Wave Resistance Ann Arbor*, pp. 819–852.

WEHAUSEN, J. V. 1969 Use of Lagrangian coordinates for ship wave resistance (first- and second-order thin-ship theory). *J. Ship Res.* **13**, 12–22.

WEHAUSEN, J. V. 1973 The wave resistance of ships. *Adv. Appl. Mech.* **13**, 93–245.

WEHAUSEN, J. V. & LAITONE, E. V. 1960 Surface waves. In *Handbuch der Physik*, vol. IX. *Fluid Dynamics*, vol. III, pp. 446–778. Springer.

YEUNG, R. W. 1972 Sinkage and trim in first-order thin-ship theory. *J. Ship Res.* **16**, 47–59.

YIM, B. 1968 Higher order wave theory of ships. *J. Ship. Res.* **12**, 237–245.